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Feedback cyclization for rings with finite stable range☆

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Abstract

For systems over commutative rings, we introduce a property called FC^s ($s > 0$) which means "feedback cyclization with s inputs": given a controllable system (A, B), there exist a feedback matrix K and a matrix U with s columns such that $(A + BK, BU)$ is controllable. Clearly, FC^1 is the usual FC property. The main result of this paper is the following: for a ring R with the GCU property (whenever (A, B) is controllable, there exists a vector u with Bu unimodular), R satisfies a strong form of the FC^s property if and only if R is s-stable, i.e. R has s in its stable range. This generalizes the known facts that 1-stable GCU rings have the FC property, and principal ideal domains, which are 2-stable GCU rings, satisfy an analogous cyclization property with two inputs. Examples are given of FC^s rings (for $s > 1$) which are not FC rings.

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1. Introduction and basic definitions

Let R be a commutative ring with 1. An m-input, n-dimensional system (or a system of size (n, m)) over R will be a pair of matrices (A, B), with $A \in R^{n\times n}$ and $B \in R^{n\times m}$. The system (A, B) is reachable or controllable if R^n is spanned by the columns of the reachability matrix $[B|AB|\cdots|A^{n-1}B]$. We say that R is an FC ring or satisfies the FC property if given a reachable system (A, B) over R, there exist a matrix K and a vector u such that $(A + BK, Bu)$ is reachable.

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A weak form of the FC property is the GCU property: if (A, B) is reachable, there exists a vector u such that Bu is unimodular, i.e. its entries generate R.

This work is motivated by the following situation. In [1, Theorem 5] Brewer, Katz and Ullery proved that a GCU ring R with 1 in its stable range has the FC property.

On the other hand, in [5, p. 272] Brewer, Klingler and Schmale observed that a principal ideal domain R satisfies the following cyclization property: if (A, B) is a reachable system, there exist a 2-column matrix U and a feedback matrix K such that $(A + BK, BU)$ is reachable. Thus, $V = \operatorname{im}(BU)$ is a rank 2 subspace of $\operatorname{im}(B)$ such that $V + (A + BK)V + \cdots + (A + BK)^{n-1}V = R^n$. We suspected that the reason why this situation is possible over a principal ideal domain is because it is a 2-stable ring [7] and a GCU ring [1], which lead us to the study of systems over GCU rings with arbitrary finite stable range conditions. We refer the reader to Estes' and Ohm's article [7] for the definition and properties of the Bass stable range, and to [1] for the discussion of many problems about systems over commutative rings.

This paper is organized as follows. In Section 2, for any positive integer s, we define the s-cyclization property in this way: a system (A, B) is s-cyclizable if there exist matrices K, U (U with s columns) such that the system $(A + BK, BU)$ is reachable. We say that R is an FC^s ring if any reachable system over R is s-cyclizable. We then prove that the FC^s property propagates to quotients and power series and lifts modulo the Jacobson radical.

In Section 3 we recall the concept of Bass stable range and give a feedback reduced form for systems over an s-stable GCU ring. This allows us to prove our main result: a GCU ring is s-stable if and only if for any reachable system (A, B) there exists an s-cyclization $(A + BK, BU)$ as above, with K, U of a special form. This generalizes some partial results known for the cases $s = 1$ and $s = 2$.

The last section contains examples and concluding remarks. We exhibit some commutative rings of arbitrary Krull dimension with the FC^s property, for some $s > 1$. When possible, we prove that our constructions cannot be improved to obtain FC^1 (=FC) rings. In applications to Control Theory, it is important to obtain the property FC^s with s as small as possible, because this allows a complete control of a reachable system with few inputs. In this paper we see that the obstruction for a ring R to solve control theory problems like feedback cyclization, appears to be the stable range of R, rather than its dimension.

2. Feedback cyclization with s inputs

We start defining a generalization of the known FC property.

Definition 2.1. A system (A, B) over a ring R is s-cyclizable ($s > 0$) if there exist matrices K, U (U with s columns), such that $(A + BK, BU)$ is reachable. Note that (A, B) is necessarily reachable, since every vector $\sum_{i=0}^{n-1}(A + BK)^i BUv_i$ is of the form $\sum_{i=0}^{n-1} A^i Bv_i'$.

We say that R is an FC^s ring if any reachable system over R is s-cyclizable. The FC^s notation is chosen because FC_s and FC-s are already used in the literature: FC_s means feedback cyclization for s-dimensional reachable systems, and FC-s means dynamic feedback cyclization with augmentation of size s.

Let us point out some remarks about this definition.

Remark 2.2. Clearly, FC^1 coincides with the usual FC property and FC^s implies $\mathrm{FC}^{s'}$ for all $s' > s$. Also, note that if in the above definition B has $m < s$ columns, then (A, B) is trivially

s-cyclizable, by taking $K = 0$ and $U = [1_m|0] \in R^{m\times s}$, i.e. the $m \times m$ identity matrix completed with $s - m$ zero columns. Finally, one-dimensional systems are s-cyclizable for all s.

We recall that two systems (A, B) and (A', B') over a ring R are feedback equivalent if $(A', B') = (PAP^{-1} + PBK, PBQ)$ for some matrices P, Q, K with P, Q invertible.

As one would expect, the s-cyclization property is invariant under feedback.

Lemma 2.3. *Let (A, B) and (A', B') be two feedback equivalent systems over a ring R. If (A', B') is s-cyclizable, then (A, B) is s-cyclizable.*

Proof. If (A', B') is s-cyclizable, there exist matrices K', U', where U' has s columns, such that the system $\Sigma_1 = (A' + B'K', B'U')$ is reachable. The equivalence $(A, B) \sim (A', B')$ means that $(A', B') = (PAP^{-1} + PBK_1, PBQ)$, for some invertible matrices P, Q and a feedback matrix K_1. But Σ_1 is feedback equivalent to the system $(P^{-1}(A' + B'K')P, P^{-1}B'U')$, which is of the form $(A + BK, BU)$, where $K = K_1P + QK'P$ and $U = QU'$ with s columns. Since reachability is invariant under feedback, this proves that (A, B) is s-cyclizable. $\square$

One consequence of the FC^s property is the following: in order to prove the FC property over an FC^s ring, it suffices to study reachable systems with $m \leqslant s$ inputs, which is known for principal ideal domains for $s = 2$. Next, we prove that the FC^s property propagates to homomorphic images and power series and lifts modulo the Jacobson radical of a ring.

Proposition 2.4. *Let R be a commutative ring with I an ideal of R and let J denote the Jacobson radical of R.*

(i) *If R has the FC^s property, then R/I has the FC^s property.*
(ii) *If R/J has the FC^s property, then R has the FC^s property.*
(iii) *R has the FC^s property if and only if $R[[x]]$ has the FC^s property.*

Proof. (i) Let (A, B) be a system over R such that its reduction $(\overline{A}, \overline{B})$ modulo I is reachable over R/I. Like in the proof of [1, Theorem 1], there exists a matrix B' over R with $\overline{B}' = \bar{0}$ such that $(A, [B|B'])$ is a reachable system over R. Since R has the FC^s property, there exist suitable matrices $K = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}$ and $U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$ such that the system $(A + BK_1 + B'K_2, BU_1 + B'U_2)$ is reachable over R. Reducing modulo I, one gets that $(\overline{A} + \overline{B}\,\overline{K}_1, \overline{B}\,\overline{U}_1)$ is a reachable system over R/I, proving that $(\overline{A}, \overline{B})$ is s-cyclizable and hence R/I has the FC^s property.

(ii) If (A, B) is a reachable system over R, then its reduction $(\overline{A}, \overline{B})$ modulo J is reachable over R/J. Therefore, there exist matrices K, U over R such that the system $(\overline{A} + \overline{B}\,\overline{K}, \overline{B}\,\overline{U})$ is reachable over R/J. But this implies that $(A + BK, BU)$ must be a reachable system over R (this is a standard application of Nakayama's lemma), proving that R is an FC^s ring.

(iii) Since the Jacobson radical of $R[[x]]$ is $J + (x)$ and one has the isomorphism $R[[x]]/(J + (x)) \cong R/J$, we apply (i) and (ii). $\square$

Now we study a strong form of the s-cyclization property.

Definition 2.5 (*Property K^s*). For a ring R and a positive integer s, consider the system (A, B) with the partition in blocks $B = [B_1|B_2]$, where $A \in R^{n\times n}$, $B_1 \in R^{n\times s}$ and $B_2 \in R^{n\times k}$. We say that (A, B) satisfies the property K^s if there exist matrices $X \in R^{k\times n}$, $Y \in R^{k\times s}$ such that $(A + B_2X, B_1 + B_2Y)$ is reachable of size (n, s). It is immediate that (A, B) is necessarily s-cyclizable, because $(A + B_2X, B_1 + B_2Y)$ is of the form $(A + BK, BU)$, for matrices $K = \begin{bmatrix} 0 \\ X \end{bmatrix}$ and $U = \begin{bmatrix} 1_s \\ Y \end{bmatrix}$. By convention, K^s holds if B has $m \leqslant s$ columns.

It is precisely this strong form of s-cyclization what will allow us to perform induction arguments afterwards. Next, we see that K^s is invariant under almost any feedback operation.

Lemma 2.6. *Let (A, B) and $(A', B') = (PAP^{-1} + PBK, PBQ)$ be two feedback equivalent systems over a ring R, with $Q = \begin{bmatrix} Q_1 & 0 \\ Q_2 & Q_3 \end{bmatrix}$, for some $Q_1 \in GL_s(R)$. If (A', B') satisfies K^s, then (A, B) satisfies K^s.*

Proof. Take $B = [B_1|B_2]$ and $B' = [B_1'|B_2']$ partitioned as in the previous definition. Operating in the equality $B' = PBQ$ we see that $B_1' = PB_1Q_1 + PB_2Q_2$ and $B_2' = PB_2Q_3$. Also, denote by K_1, K_2 the blocks of K of appropriate sizes such that $BK = B_1K_1 + B_2K_2$, so that $A' = PAP^{-1} + PB_1K_1 + PB_2K_2$.

Since (A', B') satisfies K^s, there exist matrices X', Y' such that the system $(A'', B'') = (A' + B_2'X', B_1' + B_2'Y')$ is reachable. But (A'', B'') is feedback equivalent to the reachable system $(P^{-1}A''P - P^{-1}B''Q_1^{-1}K_1P, P^{-1}B''Q_1^{-1})$, which has the form $(A + B_2X, B_1 + B_2Y)$, where $X = K_2P + Q_3X'P - Q_2Q_1^{-1}K_1P - Q_3Y'Q_1^{-1}K_1P$ and $Y = Q_2Q_1^{-1} + Q_3Y'Q_1^{-1}$. Thus, (A, B) satisfies K^s. $\square$

3. Systems over GCU rings with finite stable range

Let s be a positive integer. Following [7], we say that a ring R "has s in its stable range", or is s-stable, if given $(b_1, \ldots, b_s, b_{s+1}) = R$ there exists scalars $y_1, \ldots, y_s$ in R such that $(b_1 + y_1b_{s+1}, \ldots, b_s + y_sb_{s+1}) = R$. It is clear that if R is s-stable, then R is also s'-stable for any $s' > s$. The stable range of R is defined as the smallest s for which R is s-stable. The following matricial characterization is immediate:

Lemma 3.1. *A ring R is s-stable if and only if given row matrices $B_1 \in R^{1\times s}$, $B_2 \in R^{1\times k}$ with $[B_1|B_2]$ unimodular, there exists $Y \in R^{k\times s}$ such that $B_1 + B_2Y$ is unimodular.*

The next technical result, which is similar to [2, Lemma 2], gives a normal form for reachable systems over an s-stable GCU ring.

Lemma 3.2. *Let R be an s-stable ring with the GCU property. If a system (A, B) over R of size $(n, s + k)$ is reachable, then (A, B) is feedback equivalent to a system $(A', B') = (PAP^{-1} + PBK, PBQ)$, where*

$$A' = \begin{bmatrix} 0 & 0 \\ A_1 & A_2 \end{bmatrix}, \quad B' = \begin{bmatrix} 1 & 0 & * \\ 0 & B_1 & B_2 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} Q_1 & 0 \\ * & 1_k \end{bmatrix},$$

with $A_1 \in R^{n-1\times n-1}$, $B_1 \in R^{n-1\times s-1}$, $B_2 \in R^{n-1\times k}$, $Q_1 \in GL_s(R)$, and the remaining blocks of appropriate sizes.

Proof. If $n = 1$ and (A, B) is reachable, then B is a unimodular row of length $s + k$, hence there exists a matrix K such that $A + BK = 0$. Set $B = [B_1'|B_2']$, where $B_1' \in R^{1\times s}$ and $B_2' \in R^{1\times k}$. As R is s-stable, by Lemma 3.1 there exists a suitable Y such that the row $B_1' + B_2'Y$ is unimodular. By [2, Lemma 1], unimodular vectors over GCU rings can be completed to invertible matrices, hence there exists an $s \times s$ invertible matrix Q_1 such that $(B_1' + B_2'Y)Q_1 = [1, 0 \cdots 0]$. Now, defining $Q = \begin{bmatrix} Q_1 & 0 \\ YQ_1 & 1_k \end{bmatrix}$, it is clear that $BQ = [(B_1' + B_2'Y)Q_1|B_2']$ is equal to $[1, 0 \cdots 0|B_2']$, and $(A + BK, BQ)$ has the desired form.

If $n > 1$, by [2, Lemma 2] there exist invertible matrices P_0, Q_0 such that P_0BQ_0 is of the form $\begin{bmatrix} 1 & 0 \\ 0 & * \end{bmatrix}$, hence the first row v of P_0B is equal to the first row of Q_0^{-1} and so it must be unimodular. Applying the case $n = 1$ to v, there exists Q in the requested form such that the first row of P_0BQ is $[1, 0 \cdots 0|*]$. With further row operations (left-multiplication by an invertible matrix P_1) we obtain zeroes in the first column of P_1P_0BQ. Putting $P = P_1P_0$, we have PBQ in the desired form. Finally, a suitable feedback matrix K makes zeroes in the first row of $PAP^{-1} + PBK$, completing the proof. □

Note that although the normal form given in [2] is 'cleaner' than this one, it is crucial that any reachable system can be put into the above normal form via a feedback equivalence as in Lemma 2.6, thus preserving the presence or absence of property K^s.

We are now able to prove the main theorem of this article.

Theorem 3.3. *For a GCU ring R, the following statements are equivalent*:

(i) *Any reachable system over R of size (n, m) satisfies K^s.*
(ii) *R is s-stable.*

In particular, any s-stable GCU ring satisfies the FC^s property.

Proof. (i) ⇒ (ii): Consider a unimodular row $[B_1|B_2]$ of length $m > s$, with $B_1 \in R^{1\times s}$. The one-dimensional system $(0, [B_1|B_2])$ is reachable, so that by the property K^s there exist X, Y such that $(B_2X, B_1 + B_2Y)$ is reachable, therefore $B_1 + B_2Y$ is unimodular and R is an s-stable ring. We did not require the GCU hypothesis.

(ii) ⇒ (i): Conversely, let R be an s-stable GCU ring, and consider a reachable system (A, B) of size (n, m). We may assume $m > s$, otherwise K^s holds by convention. So, we can set $B = [B_1|B_2]$, where $B_1 \in R^{n\times s}$ and $B_2 \in R^{n\times k}$.

We proceed by induction on n. If $n = 1$, we can apply Lemma 3.1 to the unimodular row $[B_1|B_2]$: there exists a matrix Y such that the row $B_1 + B_2Y$ is unimodular, which means that the one-dimensional system $(A, B_1 + B_2Y)$ is reachable, proving that (A, B) satisfies K^s. Again, the GCU hypothesis was not necessary.

If $n > 1$, combining Lemmas 2.6 and 3.2 we can suppose (A, B) of the form:

$$\left(A = \begin{bmatrix} 0 & 0 \\ A_1 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & * \\ 0 & B_1 & B_2 \end{bmatrix} \right)$$

for some blocks $A_1 \in R^{n-1\times 1}$, $A_2 \times R^{n-1\times n-1}$, $B_1 \in R^{n-1\times s-1}$ and $B_2 \in R^{n-1\times k}$. Note that the $*$ block in the first row of B does not affect the reachability of (A, B), therefore by the well known Eising's Lemma [6] we have that the system $(A_2, [A_1|B_1|B_2])$ is reachable of size $(n - 1, m)$. Since $[A_1|B_1] \in R^{n-1\times s}$, by the induction hypothesis there exist matrices X_1 and

$[Y_1|Y_2]$ such that the system $(A_2 + B_2X_1, [A_1|B_1] + B_2[Y_1|Y_2])$ is reachable. Now consider the matrices $K \in R^{m\times n}$ and $Q \in GL_m(R)$ defined by

$$K = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ Y_1 & X_1 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1_{s-1} & 0 \\ 0 & Y_2 & 1_k \end{bmatrix}.$$

Operating, we obtain an equivalent system $(A', B') = (A + BK, BQ)$, where:

$$\begin{cases} A' = \begin{bmatrix} * & * \\ A_1 + B_2Y_1 & A_2 + B_2X_1 \end{bmatrix}, \\ B' = \begin{bmatrix} 1 & * & * \\ 0 & B_1 + B_2Y_2 & B_2 \end{bmatrix}. \end{cases}$$

By the special form of Q, Lemma 2.6 assures that (A, B) satisfies K^s if and only if (A', B') satisfies K^s. Consider the partition $B' = [B'_1|B'_2]$, with

$$B'_1 = \begin{bmatrix} 1 & * \\ 0 & B_1 + B_2Y_2 \end{bmatrix}, \qquad B'_2 = \begin{bmatrix} * \\ B_2 \end{bmatrix},$$

and note that (A', B'_1) is reachable: again, the blocks $*$ do not affect the reachability of (A', B'_1), therefore by Eising's Lemma (A', B'_1) will be reachable if and only if $(A_2 + B_2X_1, [A_1 + B_2Y_1|B_1 + B_2Y_2])$ is reachable, which is true by the induction assumption. Thus, (A', B') satisfies K^s, hence so does (A, B). The proof is complete. □

If $s = 1$, note that part (i) of the above theorem is actually equivalent to the GCU and 1-stable conditions together. Also, the case $s = 2$ explains the property for systems over principal ideal domains observed in [5]. Finally, for any s we have that the s-stable range condition is equivalent to obtaining K^s for one-dimensional reachable systems.

4. Examples and conclusions

4.1. We start proving that elementary divisor rings (possibly with zero divisors) are FC^2 rings. This was essentially known for elementary divisor domains [9, Lemma 2]. For the general case, note that such a ring R is Hermite in the sense of Kaplansky: given $a, b \in R$, there exists an invertible matrix Q such that $[a\ b]Q = [d\ 0]$. But Hermite rings are 2-stable (see [11, p. 666] for an elementary proof). Also, it follows from [1] that elementary divisor rings are GCU rings because they have the stronger property called UCU or BCU: if a matrix B has unit content, there exists a vector u with Bu unimodular. Now Theorem 3.3 implies the FC^2 property. Even one-dimensional elementary divisor rings may not be FC rings, for example $\mathbb{Z}$ is not. However, there are cases with arbitrary (even infinite) dimension which are 1-stable and hence FC rings, for example a valuation domain of arbitrary dimension. This reenforces the idea that the stable range is much more accurate than the dimension in measuring the obstruction for a ring to solve the feedback cyclization problem. If the conjecture "$\mathbb{C}[y]$ is FC" is finally proved to be true (see [5,8,9]), $\mathbb{C}[y]$ would be the first known example of an FC ring which is not 1-stable.

While working out this example, we accidentally arrived at the following result: R is an elementary divisor ring if and only if R is an Hermite ring with the UCU property. The necessity is clear, and the sufficiency follows from [1, Lemma 8] and [10, Theorem 2.1].

4.2. As a second construction, let R be a zero-dimensional ring with nilradical N, so that the Jacobson radical of $R[x]$ is $N[x]$. In the proof of [1, Theorem 2] it is shown that $R[x]/N[x] \cong (R/N)[x]$ is an elementary divisor ring. By 4.1. and Proposition 2.4(ii), $R[x]$ is an FC^2 ring. Note that $R[x]$ is not an elementary divisor ring if N is not zero [1, p. 269]. Also, $R[x]$ is not an FC ring if for some maximal ideal $\mathfrak{m}$ of R the residue field $R/\mathfrak{m}$ is finite: by Proposition 2.4(i), the FC property would propagate from $R[x]$ to the quotient $R[x]/\mathfrak{m}[x] \cong (R/\mathfrak{m})[x]$, but polynomial rings over fields with positive characteristic are not FC rings (see [8]). Another example of an FC^2 ring which is neither an elementary divisor ring nor an FC ring is $\mathbb{Z}[[x]]$.

4.3. In [4], several interesting examples are given of polynomial rings with the UCU property. Unfortunately, we did not find in the literature a unified way of computing exactly the stable range of all the rings exhibited there. For example, if V is a valuation ring (possibly with zero divisors), then $V[x]$ is a UCU ring which can have arbitrarily large dimension. As V is a local and hence a 1-stable ring, the stable range of $V[x]$ cannot be too high. For a Noetherian (discrete) valuation domain V, one has that $V[x]$ is two-dimensional and hence its stable range s is 2 or 3: $s \leqslant 3$ by [7, Theorem 2.3], and $s \geqslant 2$ because polynomial rings are never 1-stable: $(x, 1-x^2) = V[x]$ but there does not exist $k(x) \in V[x]$ such that $x + k(x)(1-x^2)$ is a unit of $V[x]$. By Theorem 3.3, $V[x]$ is an FC^s ring, which is not an FC ring if V has some finite residue field.

4.4. Let R be a Prüfer domain of dimension one or such that any nonzero element belongs to finitely many maximal ideals, for example a Dedekind domain. If R has torsion-free class group, by [3, Corollary 1] it is a GCU ring. Since all proper homomorphic images of R are 1-stable (either zero-dimensional or semi-local), this forces R to a be 2-stable ring and hence an FC^2 ring. But R may not be an FC ring (again, $\mathbb{Z}$ is an example).

4.5. If R is a noetherian ring of dimension $< d$ with the GCU property, by [7, Theorem 2.3] we have that R is d-stable and hence an FC^d ring. But if we are able to find an integer $s < d$ such that for any f in R the ring $R/(f)$ is $(s-1)$-stable, then R will be an s-stable ring and hence an FC^s ring, as was done above for $s = 2$. A deeper study of the stable range is needed to clarify this situation and yield further examples. In fact, we do not know any GCU ring which is not 2-stable. One common method of disproving the s-stable range condition is to find a unimodular vector of length $s+1$ which is not completable to an invertible matrix (see [7, Proposition 8.0]). But adding the GCU hypothesis means that stably free modules are free [2, Lemma 1], and so all unimodular vectors can be completed.

Although there remains much work to be done, our initial results show that the stable range of a ring gives an upper bound to the number of inputs sufficient to obtain cyclization for reachable systems, and thus is a useful tool to study the feedback cyclization problem.

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